Long term dynamics for nonlinear dispersive equations

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Overview and Summary

Lecture describes advances on asymptotic behavior of solutions to critical and subcritical nonlinear evolution equations.

- Linear equations with time-independent coefficients admit spectral resolution, functional calculus. Asymptotic completeness, Agmon-Kato-Kuroda (60’s, 70’s) for potentials, ongoing studies on variable metrics (trapping, nontrapping, hyperbolic trapped trajectories).

- Two types of Hamiltonian PDEs: admit “solitons” (focusing), or not (defocusing). For latter more known, expect all energy radiates to spatial infinity (scattering). Focusing equations can exhibit finite-time blowup for large data (small data: global existence, scattering).

- DKM theory for critical NLW, exterior energy estimates. Not available sub-critically for NLKG.

Linear Schrödinger equation in $\mathbb{R}^n$ with suitable decaying potential

$$i\partial_t \psi - \Delta \psi + V \psi = 0, \quad \psi(0) \in L^2(\mathbb{R}^d)$$

exhibits long-term dynamics

$$\psi(t) = \sum_j e^{itE_j} \psi_j + e^{-it\Delta} \phi_0 + o_{L^2}(1), \quad t \to \infty$$

where $(-\Delta + V)\psi_j = E_j \psi_j$, $E_j \leq 0$ are eigenfunctions, $\phi_0 \in L^2$. 

Asymptotic completeness of the wave operators 
Analogue for nonlinear equation? Soliton resolution problem.
Nonlinear analogue of asymptotic completeness?

Consider (schematic) NLS

\[ i\partial_t \psi - \Delta \psi + N(\psi) = 0, \quad \psi(0) \in H^1(\mathbb{R}^d) \]

or NLW

\[ \partial_{tt} u - \Delta u + N(u) = 0, \quad (u(0), u_t(0)) \in H^1 \times L^2(\mathbb{R}^d) \]

Assume the existence of stationary solutions \( -\Delta \phi + N(\phi) = 0 \).

Are all solutions asymptotically of the form

\[ \psi(t) = \sum_{j=1}^{J} G_j(t)(\phi_j) + \phi(t) + o(1) \quad \text{Galilei invariance} \]

\[ \vec{u}(t) = \sum_{j=1}^{J} (L_j(t)\phi_j, \dot{L}_j(t)\phi_j) + \vec{\phi}(t) + o(1) \quad \text{Lorentz invariance} \]

where the moving bulk solutions move infinitely far away from each other? May also have conformal symmetries in the critical case: scaling.
Exterior energy estimates, DKM theory

Complete answer for radial critical NLW in $\mathbb{R}^3$ by Duyckaerts-Kenig-Merle. Consider

$$\partial_{tt} u - \Delta u - u^5 = 0, \quad (u(0), u_t(0)) \in \dot{H}^1 \times L^2(\mathbb{R}^3)$$  \hspace{1cm} (1)

Global radial solutions are of the form, as $t \to \infty$ ,

$$\vec{u}(t) = \sum_{j=1}^{J} (\lambda_j(t)^{1/2} W(\lambda_j(t)x), 0) + \vec{v}(t) + o_{\dot{H}^1 \times L^2}(t)$$  \hspace{1cm} (2)

$\vec{v}$ is a free wave, and $\lambda_j$ diverge from each other. Also $\lambda_j(t) \ll t$.

Aubin-Talenti solution $W(x) = (1 + |x|^2/3)^{-1/2}$.

Note in particular that global solutions are bounded. Decoupling of the radiation from bulk terms via the following device.
$\mathbb{R}^3$ radial data, free wave $\Box u = 0$. Then for one sign $\pm$

$$\lim_{t \to \pm \infty} \int_{|x| \geq |t|} (|\nabla u|^2 + u^2_t)(t, x) \, dx \geq c \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2_t)(0, x) \, dx$$

$$\lim_{t \to \pm \infty} \int_{|x| \geq |t| + R} (|\nabla u|^2 + u^2_t)(t, x) \, dx \geq c \int_{|x| \geq R} ((ru)_r^2 + (ru_t)^2)(0, x) \, dx$$
For $R > 0$ cannot have true energy on the right-hand side: take Newton potential $u(t, x) = u(r) = r^{-1}$. Solves the wave equation in the exterior domain $|x| \geq |t| + R$. Actually,

$$\int_{|x| \geq R} (ru_r^2 + ru_t^2)(0, x) \, dr = \| \pi_{(r^{-1}, 0)}^\perp (u, u_t)(0) \|^2_{H^1 \times L^2(|x|>R)}$$

Large co-dimensions needed in higher odd dimensions. For even dimensions the exterior energy estimate fails even with $R = 0$. But does hold for radial data $(f, 0)$ in dimensions $4, 8, 12, 16, \ldots$ or $(0, g)$ in dimensions $6, 10, 14, 18, \ldots$. 
Consider radial solution $\Box u - u^5 = 0$, $\breve{u} \in \dot{H}^1 \times L^2(\mathbb{R}^3)$, $0 \leq t < T$. If

$$\breve{u} \notin \{0, (\pm W_\lambda, 0) \mid 0 < \lambda < \infty\}, \quad W_\lambda(x) = \lambda^{\frac{1}{2}} W(\lambda x)$$

where $W(x) = (1 + r^2/3)^{-\frac{1}{2}}$, then there exist $R > 0, \eta > 0$ with

$$\int_{|x| \geq R+t} \left( u_r^2(t, r) + u_t^2(t, r) \right) r^2 \, dr \geq \eta, \quad \forall \ 0 \leq t < T$$

Duyckaerts-Kenig-Merle (2012) combine this property with the concentration-compactness profile decomposition to establish soliton resolution for this equation.
Consider in $\mathbb{R}^3$ defocusing critical equation with real radial potential $V$

$$u_{tt} - \Delta u - Vu + u^5 = 0, \quad |V(x)| \lesssim \langle x \rangle^{-\sigma}, \quad \sigma > 2$$

All solutions global, for $V \geq 0$ large enough have stationary solutions. For generic $V$ there are only finitely many stationary solutions $\phi$, and no zero energy singularity of $H = -\Delta - V$ and $-\Delta - V + 5\phi^4$. Joint work with Jia-Liu-Xu 15-17:

- (2015) all radial solutions scatter to one of these.
- (2016) data that scatter to a particular stationary solution either form an open set or a $C^1$ connected manifold of finite codimension. It is a global and unique center-stable manifold, unbounded.
- (2017) nonradially, do not know soliton resolution, but retain manifold property.
Energy critical equation with potential

Construction of the manifold associated with unstable stationary solutions $\phi$ has two main parts:

- **Local construction inside a small ball.** Near any solution scattering to $(\phi, 0)$ build “center-stable” manifold of other solutions scattering to $\vec{\phi} = (\phi, 0)$ by the Lyapunov-Perron method. This manifold has *repulsivity property*, i.e., any solution which remains in the small ball for all positive times is on this local manifold. This is a perturbative argument, contraction mapping.

- **Global manifold.** Need to make sure that a solution which exits the ball in finite time cannot scatter to $\vec{\phi}$. This is not perturbative. Uses exterior energy estimates to show that any solution which is being ejected from the ball emits a small piece of radiation which pushes its energy below that of $\phi$. So cannot scatter to that solution. A form of *one-pass theorem* (Nakanishi-S.).
Reverse Strichartz

For the nonradial case the construction of the local manifold fails with standard Strichartz spaces (forbidden endpoint $L^2_t L^\infty_x(\mathbb{R}^3)$).

Lemma (Beceanu-Goldberg 2014)

Let $\Box \gamma = F$ in $\mathbb{R}^{1+3}_{t,x}$. Then

$$\|(\gamma, \gamma t)\|_{C^0_t(\dot{H}^1_x \times L^2)} + \|\gamma\|_{L^6_x t^2 \cap L^\infty_t L^2_t(\mathbb{R}^3 \times I)} \leq C \left( \|(f, g)\|_{\dot{H}^1_x \times L^2} + \|F\|_{L^{6/5}_x t^2 \cap L^{3/2}_x \cap L^3_t(\mathbb{R}^3 \times I)} \right).$$

Extends to $\Box \gamma + V \gamma = F$ provided $H = -\Delta + V$ has no zero energy eigenvalue or resonance.

Needed for $\phi^3 \eta^2$ term. Trade decay in space for decay in time.

Note that

$$\int_0^\infty \frac{1}{t} \delta(|x| - t) \, dt = \frac{1}{|x|} \in L^{3,\infty}(\mathbb{R}^3) = L^{3/2,1}(\mathbb{R}^3)^*. $$
\[ H = -\Delta + V \text{ in } L^2(\mathbb{R}^3), \text{ zero energy resonance } H\psi = 0, \text{ no zero energy eigenfunction. One has } |\psi(x)| \sim |x|^{-1}. \text{ Then} \]

**Lemma (Krieger-Nakanishi-S. 2014)**

*We have the bounds*

\[
\left\| \left( \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c - c_0 \psi \otimes \psi \right)f \right\|_{L^\infty_x L^1_t} \lesssim \|f\|_{W^{1,1}}
\]

\[
\left\| \cos(t\sqrt{H})P_c f \right\|_{L^\infty_x L^1_t} \lesssim \|f\|_{W^{2,1}}
\]

*This arises naturally in the stability analysis of } \Box u - u^5 = 0 \text{ where } H = -\Delta - 5W^4 \text{ has a zero energy resonance, namely} \]

\[ \psi = \partial_\lambda \bigg|_{\lambda = 1} W_\lambda. \]
Subcritical equations

Consider instead

\[ u_{tt} - \Delta u + u - |u|^{p-1}u = 0, \quad (u(0), u_t(0)) \in H^1 \times L^2(\mathbb{R}^3) \]

where \( 1 < p < 5 \). Stationary solutions \(-\Delta \phi + \phi - |\phi|^{p-1}\phi = 0\), need mass term for nonzero solutions to exist. However, the speed of propagation for Klein-Gordon is \( < 1 \) and so exterior energy estimate fails. Apparently difficult to separate constituents of the solution because of the varying propagation speed. Klein-Gordon displays behavior of Schrödinger for small frequencies and the wave equation for large frequencies. Little known about soliton resolution for NLS outside completely integrable equations.

Currently no DKM theorem in this context. Only partial results known, which we will describe in this talk.

A more complete picture emerges if we add dissipation.
Cubic nonlinear Klein-Gordon

Energy subcritical model equation:
\[ \square u + u = u^3 \text{ in } \mathbb{R}^{1+3}_t, x \]

\( \forall \ u(0) \in \mathcal{H} := H^1 \times L^2, \) there \( \exists! \) strong solution (Duhamel sense)
\[ u \in C^0([0, T); H^1), \ \partial_t u \in C^0([0, T); L^2) \]

for some \( T \geq T_0(\|u[0]\|_\mathcal{H}) > 0. \)

Properties: continuous dependence on data; persistence of regularity; energy conservation:
\[ E(u, \partial_t u) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} u^2 - \frac{1}{4} u^4 \right) \, dx \]

If \( \|u(0)\|_\mathcal{H} \ll 1, \) then global existence; let \( T^* > 0 \) be maximal forward time of existence:
\[ T^* < \infty \implies \|u\|_{L^3([0, T^*), L^6(\mathbb{R}^3))} = \infty \]
Basic well-posedness, focusing cubic NLKG in $\mathbb{R}^3$

If $T^* = \infty$ and $\|u\|_{L^3([0,T^*), L^6(\mathbb{R}^3))} < \infty$, then $u$ scatters: 

$\exists (\tilde{u}_0, \tilde{u}_1) \in \mathcal{H}$ s.t. for $v(t) = S_0(t)(\tilde{u}_0, \tilde{u}_1)$ one has 

$$(u(t), \partial_t u(t)) = (v(t), \partial_t v(t)) + o_{\mathcal{H}}(1) \quad t \to \infty$$

where $S_0(t)$ is the free KG evolution. If $u$ scatters, then $\|u\|_{L^3([0,\infty), L^6(\mathbb{R}^3))} < \infty$.

Finite propagation speed: if $\vec{u}(0) = 0$ on $\{|x - x_0| < R\}$, then $u(t, x) = 0$ on $\{|x - x_0| < R - t, 0 < t < \min(T^*, R)\}$.

$T > 0$, exact solution to cubic NLKG $\varphi_T(t) \sim \sqrt{2}(T - t)^{-1}$ as $t \to T_+$. Use finite prop-speed to cut off smoothly to neighborhood of cone $|x| < T - t$. Gives smooth solution to NLKG, blows up at $t = T$ or before.

By Cazenave 1985 global solutions are bounded. Not known for nonlinearities $|u|^{p{-1}}u$, with $3 < p < 5$. 

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Long term dynamics for nonlinear dispersive equations
Payne-Sattinger theorem 1975

**Small data:** global existence and scattering.  
**Large data:** can have finite time blowup.  
Is there a criterion to decide finite time blowup/global existence?  
**YES** if energy is smaller than the energy of the **ground state** \( Q \)  
unique positive, radial solution (Coffman) of:

\[
- \Delta \varphi + \varphi = \varphi^3, \quad \varphi \in H^1(\mathbb{R}^3)
\]  

(3)

Minimization problem

\[
\inf \left\{ \| \varphi \|^2_{H^1} \mid \varphi \in H^1, \| \varphi \|_4 = 1 \right\}
\]

has radial solution \( \varphi_\infty > 0 \), decays exponentially,  
\( Q = \lambda \varphi_\infty, \lambda > 0 \). Minimizes the stationary energy (or action)

\[
J(\varphi) := \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{2} \varphi^2 - \frac{1}{4} \varphi^4 \right) dx
\]

amongst all nonzero solutions of (3). Dilation functional:

\[
K_0(\varphi) = \langle J'(\varphi) | \varphi \rangle = \int_{\mathbb{R}^3} (|\nabla \varphi|^2 + \varphi^2 - |\varphi|^4)(x) dx
\]
**Theorem (Payne-Sattinger 1975)**

If \( E(u_0, u_1) < E(Q, 0) \), the dichotomy: \( K(u_0) \geq 0 \) **global existence**, \( K(u_0) < 0 \) **finite time blowup**

**Ibrahim-Masmoudi-Nakanishi (2010):** **Scattering** in addition to global existence. Requires Concentration-Compactness
Beyond Payne Sattinger in unstable case (subcritical)

**Theorem (Nakanishi-S. 2010)**

Let $E(u_0, u_1) < E(Q, 0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{rad}$. In $t \geq 0$ for NLKG:

1. **finite time blowup**
2. **global existence and scattering to 0**
3. **global existence and scattering to $Q$**:

   $$u(t) = Q + v(t) + o_{H^1}(1) \text{ as } t \to \infty, \text{ and}$$
   $$\partial_t u(t) = \partial_t v(t) + o_{L^2}(1) \text{ as } t \to \infty, \Box v + v = 0, (v, \dot{v}) \in \mathcal{H}.$$

**All 9 combinations of this trichotomy allowed as** $t \to \pm \infty$.

- Applies to all dimensions for (replace 3 by “trapped by soliton”), subcritical equations for which small data scattering is known.
- Linearized operator $L_+ = -\Delta + 1 - pQ^{p-1}$ has unique negative eigenvalue.
- Third alternative is center-stable manifold of codimension 1. Uniqueness of center-stable manifold.
The invariant manifolds

Figure: Stable, unstable, center-stable manifolds
Consider in $\mathbb{R}^d$, $d \leq 6$

$$\partial_{tt} u - \Delta u + 2\alpha \partial_t u + u - f(u) = 0$$

data $(u(0), \partial_t u(0)) \in H^1 \times L^2(\mathbb{R}^d)$, $\alpha > 0$, $f \in C^{1,\beta}(\mathbb{R})$, odd, $f'(0) = 0$, subcritical. Ambrosetti-Rabinowitz condition: there exists $\gamma > 0$ so that

$$\int_{\mathbb{R}^d} 2(1 + \gamma)F(\varphi) - \varphi f(\varphi) \leq 0 \quad \forall \varphi \in H^1(\mathbb{R}^d), \quad F' = f \quad (\ast)$$

For example

$$f(u) = \sum_{i=1}^{m_1} a_i |u|^{p_i-1} u - \sum_{j=1}^{m_2} b_j |u|^{q_j-1} u , \quad 1 < q_j < p_i \leq \frac{d+2}{d-2}, \quad \forall i, j \quad (\dagger)$$

$$a_i, b_j \geq 0, a_{m_1} > 0 .$$

For this class existence, uniqueness of ground state known, hyperbolicity of linearized operator around $(Q,0)$. Infinitely many radial stationary solutions (Berestycki-Lions 83). We only assume $(\ast)$ not $(\dagger)$. 

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Convergence to equilibria or blowup

**Theorem (Burq-Raugel-S ’15)**

Let \( \alpha > 0 \). Assume that \( 1 \leq d \leq 6 \) and that nonlinearity satisfies above conditions. Then any solution with radial \( H^1 \times L^2 \) data

1. either blows up in finite time,
2. or exists globally and **converges** exponentially to an equilibrium point (stationary solution) as \( t \to +\infty \).

Does not use concentration-compactness, but relies heavily on results from dynamical systems in infinite dimensions (invariant manifold theory, Chen-Hale-Tan, Brunovsky-Polacik 90s).

Energy is monotone decreasing:

\[
E(\bar{u}(T)) - E(\bar{u}(0)) = -2\alpha \int_0^T \| u_t(t) \|_{L^2}^2 \, dt
\]

Implies: \( \omega \)-limit set of any global solution consists of equilibria (stationary solutions), or empty.
Convergence to equilibria or blowup: scheme of proof

- Not clear a priori if global solution is bounded in $H^1 \times L^2$.
- Let $K_0(\varphi) = \int_{\mathbb{R}^d} |\nabla \varphi|^2 + \varphi^2 - \varphi f(\varphi) \, dx$. Show $\exists t_n \to \infty$ s.t. $K_0(u(t_n)) \to 0$, $\int_{t_{n-1}}^{t_{n+1}} \|u_t\|_{L^2} \to 0$.
- Then show that $\vec{u}(t_n) \to (Q, 0)$, a stationary solution.
- Linearize about $(Q, 0)$. We may or may not have hyperbolicity of the linearized equation, depends on whether $L_Q := -\Delta + 1 - f'(Q)$ has trivial kernel or not; in latter case kernel is 1-dimensional (due to radial assumption).
- Construct stable, unstable, center(un)stable manifolds near $(Q, 0)$ (Chen-Hale-Tan 1997). Latter only present if $L_Q$ has nontrivial kernel. If present, center manifold is a curve.
- Now apply Brunovsky-Polacik (97): if center dynamics is stable, $\vec{u}(t) \not\to (Q, 0)$ as $t \to \infty$ implies $\vec{u}(\tilde{t}_n) \to (\tilde{Q}, 0) \neq (Q, 0)$ which belongs to unstable manifold. But such an equilibrium cannot lie on unstable manifold, so done. Stability of center manifold: it is a curve, and infinitely many equilibria on it. So evolution is trapped between them.
The spectrum of the linearized flow with dissipation

Figure: The spectrum of the damped equation, $0 < \alpha < 1$. 
Some details of proof

One has

$$\gamma(\|\phi\|_{H^1}^2 + \|\psi\|_{L^2}^2) \leq 2(1 + \gamma)E(\phi, \psi) - K_0(\phi)$$

So $K_0(u(t)) \geq -M$ on interval of existence implies solution global. Define

$$y(t) = \frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t \|u(s)\|_{L^2}^2 ds$$

Then

$$\ddot{y}(t) = \|\partial_t u(t)\|_{L^2}^2 - K_0(u(t)) \quad (\star)$$

If $K_0(u(t)) \leq -\delta$ for large times, then by strict convexity $y(t) \to \infty$ (assume global solution). If $\alpha = 0$ then deduce via energy that

$$\ddot{y}(t) \geq \frac{2 + \gamma \dot{y}(t)^2}{\gamma y(t)} \quad \text{or} \quad \frac{d^2}{dt^2}(y^{\gamma/2}(t)) < 0$$

So finite time blowup. If $\alpha > 0$ more work needed.
Suppose \( u(t) \) global solution (and so energy remains positive). Cannot have \( K_0(u(t)) \leq -\kappa < 0 \) for large times. Also cannot have \( K_0(u(t)) \geq \kappa > 0 \) for large times: (i) solution is bounded (ii) (*) implies that

\[
\dot{y}(t_2) - \dot{y}(t_1) \leq \int_{t_1}^{t_2} \| \partial_t u(t) \|^2_{L^2} \, dt - (t_2 - t_1) \kappa
\]

Thus, \( K_0(u(t_n)) \to 0 \) for some sequence \( t_n \to \infty \). Thus, \( \| \tilde{u}(t_n) \|_H \) uniformly bounded, \( \int_{t_n-1}^{t_n+1} \| \partial_t u(t) \|^2_{L^2} \, dt \to 0 \) and \( \tilde{u}_n(s) := \tilde{u}(t_n + s) \), \(-1 \leq s \leq 1\) converges to \( \tilde{u}^* = (Q, 0) \) (equilibrium). How to obtain strong convergence in \( H^1 \): (i) \( u_n(0) \rightharpoonup u^* \) in \( H^1 \) (ii) \( K_0(u^*) = 0 \) by equilibrium (iii) \( K_0(u_n(0)) \to 0 \) (iv) Thus \( \| u_n(0) \|_{H^1} \to \| u^* \|_{H^1} \) (use compact radial Rellich embedding on nonlinear term) (v) strong convergence. More work needed to prove that \( \| \partial_t u_n(0) \|_{L^2} \to 0 \).
Time dependent asymptotically vanishing damping

Consider in $\mathbb{R}^d$, $d \leq 6$

$$\partial_{tt}u - \Delta u + 2\alpha(t)\partial_t u + u - f(u) = 0$$

We assume $\alpha(t) > 0$, $\int_0^\infty \alpha(t)\,dt = \infty$. In fact, $\alpha(t) = (1 + t)^{-a}$, $0 < a < \frac{1}{3}$. Let $f(u)$ be as above.

**Theorem (Burq-Raugel-S., 18):** Any solution with radial $H^1 \times L^2$ data

1. either blows up in finite time,
2. or exists globally and converges to an equilibrium point (stationary solution) as $t \to +\infty$.

Proof does not use invariant manifolds (non-autonomous case, asymptotically vanishing damping, so manifolds difficult to use). Rather, it relies on functional approach, Łojasiewicz-Simon inequality. Nonlinearity $f(u) = |u|^{p-1}u$, case $d = 3$ and $4 < p < 5$ more delicate, requires more PDE techniques.
$F : \Omega \to \mathbb{R}$ real-analytic on some domain $\Omega \subset \mathbb{R}^n$, $\nabla F(a) = 0$. There exists $1 < \theta \leq 2$ so that

$$|F(x) - F(a)| \leq C|\nabla F(x)|^\theta, \quad \forall |x - a| < \varepsilon \quad (\star)$$

Consider ODE

$$\dot{u}(t) + \nabla F(u(t)) = 0, \quad u(0) = u_0 \in \mathbb{R}^n$$

If trajectory global and bounded, then $\omega$ limit set is exactly one point. Idea: $u(t_n) \to p$ along some sequence going to $\infty$. Consider Lyapunov functional with $\theta = 2, a = p$ in $(\star)$

$$G(u) := F(u) - F(p)$$

Then $\frac{d}{dt} G(u(t)) = -|\nabla F(u(t))|^2 = -|\dot{u}(t)|^2 \leq -cG(u(t))$. Exponential decrease and convergence.
Back to PDE: Subcriticality and a key ingredient

**Subcritical lemma:** Let \( \vec{u} \) be a global trajectory and suppose that for some equilibrium \( Q \) and for some time \( t_0 \)

\[
\| \vec{u}(t_0) - (Q, 0) \|_H < \rho,
\]

where \( 0 < \rho < 1 \) is arbitrary but fixed. There exists a small \( \delta_0 = \delta_0(\rho, d, Q, f) \) (independent of \( \alpha \)) with the following property: if \( \| u(t) - Q \|_{L^2} \leq \delta_0 \) for all \( t \in [t_0, t_1] \) where \( t_1 \geq t_0 \) is arbitrary, then

\[
\| \vec{u}(t) - (Q, 0) \|_H \leq 2\rho
\]

for all \( t \in [t_0, t_1] \).

Holds also for \( \alpha = 0 \). We use damping to obtain finiteness and smallness of expressions of the form

\[
\int_I \| u_t(t) \|_{L^2} \, dt
\]

for sufficiently long \( I \). In fact, we need this with \( I = [t_*, \infty) \).
Local integrability and integrability of $\| u_t \|_{L^2}$

Let $\vec{u}(t)$ be a global trajectory. Then energy remains positive, so

$$E(\vec{u}(0)) - E(\vec{u}(\infty)) = \int_0^\infty \alpha(t) \| u_t(t) \|_{L^2}^2 dt < \infty$$

Suppose $I_n = [b_n, b_{n+1})$ satisfies

$$\int_{I_n} \| u_t(t) \|_{L^2}^2 dt \geq \epsilon > 0 \quad \forall \ n$$

Then

$$\int_0^\infty \alpha(t) \| u_t(t) \|_{L^2}^2 dt \geq \sum_n \alpha(b_{n+1}) \int_{I_n} \| u_t(t) \|_{L^2}^2 dt \geq \epsilon \sum_n b_{n+1}^{-a} = \infty$$

if $b_n$ do not grow too quickly. In fact along some subsequence of integers

$$\int_{n^{\frac{3}{2}}}^{(n+1)^{\frac{3}{2}}} s^\frac{1}{3} \| u_t(s) \|_{L^2}^2 ds \to 0$$
Main functional

By Cauchy-Schwarz obtain

\[
\int_{n^\frac{3}{2}}^{(n+1)^\frac{3}{2}} \|u_t(s)\|_{L^2} \, ds \to 0
\]

As for constant damping conclude \(\|\vec{u}(t_n) - (Q, 0)\|_H \to 0\) for some sequence \(t_n \in [n^\frac{3}{2}, n^\frac{3}{2} + 1]\). With subcritical lemma conclude that

\[
\max_{l_n} \|\vec{u}(s) - (Q, 0)\|_H \to 0, \quad l_n = [n^\frac{3}{2} + 1, (n + 1)^\frac{3}{2}]
\]

Now want to upgrade this to full convergence. Based on delicate analysis of functional with \(\nu = 1+\)

\[
H_\nu(t) = E(\vec{u}(t)) - E(Q, 0) + \frac{\varepsilon_0}{(1 + t)^{\alpha \nu}} \langle -\Delta u + u - f(u), u_t \rangle_{H^{-1}}
\]

Motivated by work of L. Simon, Haraux-Jendoubi on gradient systems.
Main bootstrap strategy

Take a time $t_0 \in I_n$ for $n$ large. Let $t_0^* > t_0$ be maximal with

$$\| \vec{u}(t) - (Q, 0) \|_\mathcal{H} < \rho_0, \quad \forall \ t_0 \leq t_0^*$$

Then deduce from monotonicity and decay properties of $H_\nu(t)$ that in fact

$$\| \vec{u}(t) - (Q, 0) \|_\mathcal{H} < \rho_0/2, \quad \forall \ t_0 \leq t_0^*$$

whence $t_0^* = \infty$. This is accomplished by showing that for some $\eta > 0$ one has

$$\int_{t_0}^{T_1} \| u_t(t) \|_2 \ dt \leq t_0^{-\eta} \quad (\ast)$$

where $T_1 > t_0$ maximal with $\| \vec{u}(t) \|_\mathcal{H} \leq R_1$. The quantitative control $(\ast)$ is obtained by integrating $H_\nu(t)$, see the next slide. Requires $H_\nu(t) > 0$ and decay of $H_\nu$. 
Łojasiewicz-Simon inequality

On intervals $t_0 \leq t \leq t_1$ with $\|\vec{u}(t)\|_{H^1 \times L^2} \leq R_1$ one has $H_\nu$ is decreasing (however, $\dim = 3, 4 < p < 5$ more complicated, requires Strichartz estimates). Interplay between $H_\nu, \dot{H}_\nu$, stationary energy $J$:

$$
\dot{H}_\nu(t) \leq -\frac{2 - \varepsilon_0 C(R_1)}{(1 + t)^a} \|u_t\|_{L^2}^2 - \frac{\varepsilon_0}{4(1 + t)^{a\nu}} \|J'(u)\|_{H^{-1}}^2
$$

$$
J'(u) = -\Delta u + u - f(u)
$$

Two possibilities:
- $H_\nu > 0$ on long subinterval of $t_0 \leq t \leq t_1$
- $H_\nu \leq 0$ on long subinterval of $t_0 \leq t \leq t_1$

**Case 1:** If $H_\nu(t) > 0$ use radial Łojasiewicz-Simon (L.-S.) inequality

$$
|J(u) - J(Q)| \leq C\|J'(u)\|_{H^{-1}}^2, \quad \|u - Q\|_{H^1} \ll 1
$$

Intuitively, obtained by Taylor expansion. More subtle due to possible lack of invertibility of second variation.
Łojasiewicz-Simon inequality for $H_\nu > 0$

Easy to prove if linearization (second variation) $-\Delta + 1 - f'(Q)$ has trivial kernel. In the radial setting, the dimension of the kernel is at most 1. Combining the expressions for $H_\nu$, $\dot{H}_\nu$ via the Łojasiewicz-Simon inequality implies that

$$\dot{H}_\nu(t) + \frac{C_2 \varepsilon_0}{(1 + t)^{a\nu}} H_\nu(t) \leq 0$$

Only useful if $H_\nu > 0$, gives decay of $H_\nu$ over sufficiently long intervals. Combine with $\dot{H}_\nu$ to obtain crucial integrability of $\|u_t\|_{L^2}$.

Conclude that if $\|\bar{u}(t) - (Q, 0)\|_{H^1 \times L^2} < \rho_0$ on some long time interval $I$, then the trajectory $\bar{u}(t)$ stays close to $(Q, 0)$ forever, and approaches $(Q, 0)$ at the rate

$$\exp\left(-Ct^{1-a}\right)$$
Case 2: If $H_{\nu} < 0$ use

$$0 < E(\bar{u}(t)) - E(Q,0) \leq \frac{2\varepsilon_0}{(1 + t)^{a(\nu+1)}} \|u_t\|_{H^{-1}}^2 + \frac{\varepsilon_0}{2(1 + t)^{a\nu}} \partial_t \|u_t\|_{H^{-1}}^2$$

$$0 < E(\bar{u}(t)) - E(Q,0) \leq \frac{\varepsilon_0 C_1}{(1 + t)^{a\nu}} \|u_t\|_{H^{-1}} \leq \frac{\varepsilon_0 C_1}{(1 + t)^{a\nu}} \|u_t\|_{L^2}$$

Proposition: Assume $\exists \; t_1 \in I_m$ s.t. $H_{\nu}(t_1) < 0$. Let $[t_1, T_1]$ be maximal interval with $\|\bar{u}(t)\|_{\mathcal{H}} \leq R_1$. Then $H_{\nu}(t)$ decreasing, $H_{\nu}(t) < 0$ on $[t_1, T_1]$. If $m$ large, $\nu - 1 > 0$ small,

$$\int_{t_1}^{T_1} \|u_t(s)\|_{L^2} \, ds \leq C t_1^{-a(\nu-1)}, \quad (4)$$

where $C > 0$ is an absolute constant.

Conclusion by contradiction: from (4) by maximality $T_1 = \infty$ and thus $\|\bar{u}(t) - (Q,0)\|_{H^1 \times L^2} \leq \rho_0$. So in fact this case cannot occur.
We can no longer control $\dot{H}_\nu(t)$ pointwise in time due to term $\langle f'(u)u_t, u_t \rangle_{H^{-1}}$. Requires averaging in time, Strichartz estimates, and observability inequality:

$$\|u_t\|_{L^\infty(I,L^2(\mathbb{R}^d))} \leq C(I)\|u_t\|_{L^2(I,L^2(\mathbb{R}^d))}$$  \tag{1}

for any finite time interval $I$. This holds for the (damped) NKLG. For free Klein-Gordon one has the stronger

$$\|\tilde{u}\|_{L^\infty(I,H(\mathbb{R}^d))} \leq C(I)\|u_t\|_{L^2(I,L^2(\mathbb{R}^d))}$$

Stationary solutions show that only the weaker (1) is possible.

We have monotonicity $H_\nu(t + \tau_0) \leq H_\nu(t)$, i.e., along arithmetic progressions. This suffices for the method.